

Stochastic Properties of a One-Dimensional Discrete Ginzburg-Landau Field

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Starting from a master equation for a discrete order parameter a dynamical model is set up via mean-field approximation in the Fokker-Planck equation. The time evolution of some mean values is calculated numerically, showing two transitions with characteristic slowing down of the relaxation time.

The widespread use of the Ginzburg-Landau model in equilibrium statistics supports growing interest in the dynamical behaviour of the model. For the purposes of thermodynamics a free energy functional is minimized with respect to an order parameter field, or, in a generalization to include fluctuations, a functional integration [1] over this field is performed to get the thermodynamic averages. The investigation of stochastic properties may start from a master equation, reflecting the Chapman-Kolmogorov equation of a Markov process. The latter is assumed for simplicity and for generality in replacing the complicated dynamics of microscopic variables by a stochastic process of macrovariables [2], i.e. of the order parameter defined on a space divided into cells of linear dimension which is great compared with the interparticle distance and small compared with the length scale of macroscopic variation. An ansatz for the transition rates consistent with the known equilibrium solution of the master equation is needed and is got only with some intuition leaving a great deal of indefiniteness. The equilibrium solution should be of course the Einstein probability with the Ginzburg-Landau free energy.

Because of the lower bound for the cell dimension one is forced to a discrete version of the Ginzburg-Landau theory if macrovariables are introduced. This version reveals a close similarity to a spin model suggesting a master equation approach analogous to the kinetic Ising model, which has been extensively studied [3], [4]. The relation becomes obvious if only a discrete set of real values is as-

signed to the order parameter which itself is defined on a discrete space. The space derivative part in the free energy functional, $|\nabla\psi|^2$, then becomes equivalent to a spin hamiltonian with nearest neighbour interaction

$$\sum_j |\mathbf{R}_j - \mathbf{R}_i|^{-2} |\psi(\mathbf{R}_j) - \psi(\mathbf{R}_i)|^2$$

conceiving the order parameter as a dynamical variable. Physically speaking the stiffness of the order parameter against a variation in space is reflected by a minimum energy contribution for equal values in neighbouring cells. From this point of view the discrete model formally reveals two competing mechanisms which may drive the system into a critical behaviour: the first is represented by the quadratic and quartic term, $a\psi^2 + b\psi^4$, which constitute the total free energy if the order parameter is forced to be constant, the second emerges from the spin-like interaction part. In superconductivity the first energy contribution accounts for the usual phase transition at $T = T_c$ where the coefficient a vanishes. The importance of the second is seen in type-II-superconductors where the external field prevents the order parameter from being constant in the intermediate phase and a further phase transition into the Meissner phase with constant order parameter occurs on lowering the field.

The discrete model fits well into the picture of granular superconductors which show two successive phase transitions. First the individual grains become superconducting with different values of the order parameter on different grains. At lower temperatures an additional ordering is observed which is attributed to the interaction between the grains via the Josephson coupling [5].

The outlined picture will be studied in the simplest approach, which comprises both kinds of

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transitions. First we confine ourselves to a one-dimensional system of a linear chain consisting of discrete segments. This prohibits a strict phase transition, however, a nearly critical behaviour may be detected by a small relaxation rate, by great fluctuations or by a considerable slope in the temperature dependence of the thermodynamic functions. Besides, we restrict the values of the order parameter to a set of three distinct values. A number of three is necessary to distinguish between the quadratic and quartic term in the free energy. The model may be visualized as a one-dimensional granular superconductor, though we consider the interaction between neighbouring segments only via a non-negative order parameter which represents a surface energy but neglects the phase coupling.

Instead of treating the master equation we derive the Fokker-Planck equation in two sum variables, the mean order parameter and its mean square deviation. To this end the transition rates have to be chosen as functions of these variables alone, which corresponds to the mean-field approximation in equilibrium. In the interaction the order parameter of both neighbouring cells is replaced by its mean value which of course is still a stochastic variable. On the one hand in equilibrium statistics this is an antiquated approximation commonly recognized as poor. On the other hand away from the equilibrium its effect changes, because the mean value itself fluctuates. Anyway, it has to be kept in mind that the approximation overestimates the

interaction and is rather modelling a system with either long-range forces or randomly moving particles interacting with each other. The latter corresponds e.g. to the population model of Weidlich [4], which is thereby generalized to include a third state as an alternative “opinion”.

The details of the model will be presented in the following two sections, where the first deals with the derivation of a two-dimensional Fokker-Planck equation and the second specifies the transition rates. In a last section the time evolution of the averaged order parameter and its fluctuation along the chain are determined. For comparison the equilibrium quantities as e.g. the relaxation rates are shown, too.

Fokker-Planck Equation

We consider a cyclic closed chain of N segments of length Δx which may be thought as small superconducting strips. The state of the i -th segment is characterized by an order parameter ψ_i with values 0, $\frac{1}{2}$ and 1 for the normal conducting, an intermediate and the superconducting state, respectively, according to the gap of superconductivity. Denoting $\tilde{P}(\psi_1 \dots \psi_N; t)$ the probability that at time t the state of the chain with configuration $\psi_1 \dots \psi_N$ is realized and by $\tilde{w}_{\psi_i \rightarrow \psi_i'}^{(i)}(\psi_1 \dots \psi_N)$ the rate for a transition of the i -th segment from state ψ_i to ψ_i' with initial configuration $\psi_1 \dots \psi_N$ we get the master equation

$$\begin{aligned} \partial_t \tilde{P}(\psi_1 \dots \psi_N; t) = & \sum_{i=1}^N \{ \tilde{w}_{\psi_i + \frac{1}{2} \rightarrow \psi_i}^{(i)}(\psi_1 \dots \psi_i + \frac{1}{2} \dots \psi_N) \tilde{P}(\psi_1 \dots \psi_i + \frac{1}{2} \dots \psi_N; t) \\ & + \tilde{w}_{\psi_i - \frac{1}{2} \rightarrow \psi_i}^{(i)}(\dots \psi_i - \frac{1}{2} \dots) \tilde{P}(\dots \psi_i - \frac{1}{2} \dots; t) \\ & - \tilde{w}_{\psi_i \rightarrow \psi_i + \frac{1}{2}}^{(i)}(\dots \psi_i \dots) \tilde{P}(\dots \psi_i \dots; t) - \tilde{w}_{\psi_i \rightarrow \psi_i - \frac{1}{2}}^{(i)}(\dots \psi_i \dots) \tilde{P}(\dots \psi_i \dots; t) \}. \quad (1) \end{aligned}$$

Direct transitions $\psi_i = 0 \leftrightarrow \psi_i = 1$ are excluded, i.e. a change of the segment state always goes through the intermediate state. The transition rate vanishes for an order parameter outside the assigned interval, $\tilde{w}_{\psi_i \rightarrow \psi_i + \frac{1}{2}}^{(i)} = 0$ for $\psi_i = 1$ etc. To derive the Fokker-Planck equation suitable global variables have to be specified. We take the number of segments in each of the possible states as such variables

$$n_1 = \sum_{\{\psi_i = 0\}} 1, \quad n_2 = \sum_{\{\psi_i = \frac{1}{2}\}} 1, \quad n_3 = \sum_{\{\psi_i = 1\}} 1 \quad (2)$$

with the restriction $n_1 + n_2 + n_3 = N$. This implies an approximation for the transition rates which

now should only depend on the occupation numbers n_i and not on the special configuration. A multinomial factor which counts the configurations compatible with one set n_1, n_2, n_3 is necessary in order that the probability is properly normalized over the allowed values of n_i .

$$P(n_1, n_2, n_3; t) \equiv \frac{N!}{n_1! n_2! n_3!} \cdot \tilde{P}(\psi_1 \dots \psi_N; t). \quad (3)$$

The resulting master equation may be expanded in a Kramers-Moyal expansion up to the first order

in $1/N$

$$\begin{aligned} & \partial_t P(x_1 x_2 x_3; t) \\ &= \sum_{(\alpha\beta)=12;21;23;32} \left\{ (\partial_\alpha - \partial_\beta) + \frac{1}{2N} (\partial_\alpha - \partial_\beta)^2 \right\} \\ & \quad \cdot x_\alpha w_{\alpha\beta}(x_1 x_2 x_3) P(x_1 x_2 x_3; t) \end{aligned} \quad (4)$$

with $x_i = n_i/N$. The label at the transition rate $\alpha = 1, 2, 3$ denotes the values of the order parameter $\psi = 0, \frac{1}{2}, 1$ respectively, so that w_{12} describes the transition from $\psi = 0$ to $\psi = \frac{1}{2}$ regardless of the special position of the segment which will switch. The time dependence of the averaged occupation numbers follows from the definition

$$\begin{aligned} \bar{x}_\alpha(t) &= \int_{\mathcal{D}} x_\alpha P(x_1 x_2 x_3; t) dv, \\ dv &= dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1) \end{aligned} \quad (5)$$

by applying Eq. (4) with suitable boundary conditions yielding

$$\frac{d\bar{x}_\gamma}{dt} = \sum_{\mathcal{D}} \int (x_\alpha w_{\alpha\gamma} - x_\gamma w_{\gamma\alpha}) P dv, \quad (6)$$

where \mathcal{D} is the domain $0 \leq x_\alpha \leq 1$. Equation (6) is also directly obtained without any assumption about the boundary values from the unexpanded form leading to Equation (4). The boundary conditions for Eq. (4) are not unambiguously defined. However, the following choice guarantees the time independence of the normalization of probability.

$$\begin{aligned} x_1 = 0: & \left(1 + \frac{1}{2N} (\partial_2 - \partial_1) \right) x_2 w_{21} P = \frac{1}{2N} w_{12} P. \\ x_3 = 0: & \left(1 + \frac{1}{2N} (\partial_2 - \partial_3) \right) x_2 w_{23} P = \frac{1}{2N} w_{32} P, \\ x_2 = 0: & \left[\left(1 + \frac{1}{2N} (\partial_1 - \partial_2) \right) x_1 w_{12} \right. \\ & \quad \left. + \left(1 + \frac{1}{2N} (\partial_3 - \partial_2) \right) x_3 w_{32} \right] P \\ &= \frac{1}{2N} (w_{21} + w_{23}) P. \end{aligned} \quad (7)$$

Only two variables are needed to describe the system and we drop $x_2 (= 1 - x_3 - x_1)$ introducing $u = -x_1$, $v = x_3$ which yields

$$\begin{aligned} \partial_t P &= -\partial_u \left(a_- - \frac{1}{2N} \partial_u a_+ \right) P \\ & \quad - \partial_v \left(b_- - \frac{1}{2N} \partial_v b_+ \right) P \end{aligned} \quad (8)$$

with

$$\begin{aligned} a_\pm &= -u w_{12} \pm (1 + u - v) w_{21}, \\ b_\pm &= (1 + u - v) w_{23} \pm v w_{32}. \end{aligned} \quad (9)$$

We consider the time dependence of some averaged quantities, i.e. the mean order parameter

$$\bar{x} \equiv -\frac{1}{2} (\bar{x}_1 - \bar{x}_3) = \frac{1}{N} \sum_{i=1}^N \left(\bar{\psi}_i - \frac{1}{2} \right) \quad (10)$$

and its mean square

$$\bar{y} \equiv \frac{1}{4} (\bar{x}_1 + \bar{x}_3) = \frac{1}{N} \sum_{i=1}^N \left(\bar{\psi}_i - \frac{1}{2} \right)^2, \quad (11)$$

where the right-hand part of each equation follows with Equation (2). The corresponding equations of motion are

$$d\bar{x}/dt = \frac{1}{2} (a_-(\bar{x}, \bar{y}) + b_-(\bar{x}, \bar{y})), \quad (12)$$

$$d\bar{y}/dt = -\frac{1}{4} (a_-(\bar{x}, \bar{y}) - b_-(\bar{x}, \bar{y})). \quad (13)$$

Thereby it is assumed that the probability distribution is strongly peaked allowing an approximation which replaces the average of a function by the function of the average, e.g. $\overline{a_-(x, y)} = a_-(\bar{x}, \bar{y})$. This fails near critical fluctuations where the probability is rather flat. Nevertheless it helps to detect the instabilities and provides an impression of the time development. The assumption is good as long as the Gauß approximation [6] to a solution of Eq. (8) is justified. The latter is obtained by approximating the coefficients a_- , b_- by a linear function and a_+ , b_+ by constants with respect to u , v . Near the instabilities the derivatives $\partial_u a_-$, $\partial_v b_-$ vanish and the Gauß approximation is no longer valid as well.

The transition rates determined in the following section are based on the condition of detailed balance for the transitions in Eq. (1) implying trivially that the equilibrium distribution is a stationary solution also of the derived equations, e.g. the Fokker-Planck equation within the limits of validity of the used approximation, i.e. to order $1/N^2$. Thus anticipating the results of the next section we have a stationary solution of Eq. (8)

$$P_e(u, v) = C \exp \left\{ -\frac{\beta \Delta x}{\xi_0} \sum_{i=1}^N f(\psi_{i+1}, \psi_i) \right\}$$

with

$$C = \frac{\text{const}}{\Gamma(1 - Nu) \Gamma(1 + Nv) \Gamma(1 + N(1 + u - v))},$$

$$\sum_{i=1}^N f(\psi_{i+1} \psi_i) = N \left\{ \left(\frac{\tilde{a}}{4} + \frac{\tilde{c}}{2(\Delta x)^2} \right) (1 + u + 3v) + \frac{\tilde{b}}{16} (1 + u + 15v) - \frac{\tilde{c}}{2(\Delta x)^2} (1 + u + v)^2 \right\}$$

which is easily shown to give zero result in each of both contributions to the right-hand side of Equation (8). Therefore

$$\left(a_- - \frac{1}{2N} \partial_u a_+ \right) P = \text{const},$$

$$\left(b_- - \frac{1}{2N} \partial_v b_+ \right) P = \text{const}$$

up to order $1/N^2$ reflecting the existence of a potential of the Fokker-Planck equation [7]. Within the same accuracy the boundary conditions, Eqs. (7), are satisfied.

Transition Rates

To exploit the condition of detailed balance the equilibrium distribution has to be specified. The

$$\sum_{i=1}^N f(\psi_{i+1}, \psi_i) = \sum_{i=1}^N \left\{ \tilde{a} \psi_i^2 + \tilde{b} \psi_i^4 + \frac{2\tilde{c}}{(\Delta x)^2} \left(\psi_i^2 - 2 \left(x + \frac{1}{2} \right) \psi_i + \left(x + \frac{1}{2} \right)^2 \right) \right\}. \quad (18)$$

Detailed balance implies to leading order in $1/N$

$$\frac{\tilde{w}_{\psi_k \rightarrow \psi_k}^{(k)}}{\tilde{w}_{\psi_k \rightarrow \psi_k'}^{(k)}} = \frac{e^{-\beta F(\psi_1 \dots \psi_k \dots \psi_N)}}{e^{-\beta F(\psi_1 \dots \psi_k' \dots \psi_N)}} = \exp \left\{ -\frac{\beta \Delta x}{\xi_0} \left[\left(\tilde{a} + \frac{2\tilde{c}}{(\Delta x)^2} \right) (\psi_k^2 - \psi_k'^2) + \tilde{b} (\psi_k^4 - \psi_k'^4) - \frac{4\tilde{c}}{(\Delta x)^2} \left(x + \frac{1}{2} \right) (\psi_k - \psi_k') \right] \right\} \quad (19)$$

especially

$$w_{12}/w_{21} = \exp \left\{ -\frac{\beta \Delta x}{\xi_0} \left[\frac{\tilde{a}}{4} + \frac{\tilde{b}}{16} - \frac{2\tilde{c}}{(\Delta x)^2} \left(x + \frac{1}{4} \right) \right] \right\},$$

$$w_{23}/w_{32} = \exp \left\{ -\frac{\beta \Delta x}{\xi_0} \left[\frac{3\tilde{a}}{4} + \frac{15\tilde{b}}{16} - \frac{2\tilde{c}}{(\Delta x)^2} \left(x - \frac{1}{4} \right) \right] \right\}. \quad (20)$$

It suggests the following ansatz, an exponential function with exponent being linear in x

$$w_{12} \sim \exp \left\{ \frac{\beta \Delta x}{\xi_0} \left[\beta_{12} + \alpha_{12} \frac{2\tilde{c}}{(\Delta x)^2} \left(x + \frac{1}{2} \right) \right] \right\},$$

$$w_{32} \sim \exp \left\{ \frac{\beta \Delta x}{\xi_0} \left[-\beta_{32} + \alpha_{32} \frac{2\tilde{c}}{(\Delta x)^2} \left(\frac{1}{2} - x \right) \right] \right\},$$

$$w_{21} \sim \exp \left\{ \frac{\beta \Delta x}{\xi_0} \left[-\beta_{21} - \alpha_{21} \frac{2\tilde{c}}{(\Delta x)^2} x \right] \right\},$$

$$w_{23} \sim \exp \left\{ \frac{\beta \Delta x}{\xi_0} \left[\beta_{23} + \alpha_{23} \frac{2\tilde{c}}{(\Delta x)^2} x \right] \right\}. \quad (21)$$

Einstein probability can be written as

$$P_e(\psi) \sim e^{-\beta F(\psi)}, \quad (14)$$

where $F(\psi)$ is the Ginzburg-Landau free energy functional as given by Scalapino et al. [1]

$$F(\psi) = \int_0^L \frac{dx}{\xi_0} \left[\tilde{a} \psi^2 + \tilde{b} \psi^4 + \tilde{c} \left(\frac{d\psi}{dx} \right)^2 \right]. \quad (15)$$

For a discrete chain we have

$$F(\psi_1 \dots \psi_N) = \frac{\Delta x}{\xi_0} \sum_{i=1}^N f(\psi_{i+1}, \psi_i) \quad (16)$$

with length Δx of a single segment. The mean-field approximation incorporates

$$\sum_{i=1}^N (\psi_{i+1} - \psi_i)^2 \approx 2 \sum_i \left(\frac{1}{N} \sum_{l=1}^N \psi_l - \psi_i \right)^2 \quad (17)$$

and leads to

In superconductivity the expression with coefficient \tilde{c} leads to a surface energy between two neighbouring segments, whereas the remaining part reflects the volume energy of a single segment. According to this interpretation the force exerted by neighbours in state $\psi_k = \frac{1}{2}$ or 1 on a segment in state $\psi_i = 0$ favours transitions to $\psi_i = \frac{1}{2}$ with an energy contribution proportional to the number of segments being in state 1 plus one half the number of segments being in state $\frac{1}{2}$, on the whole

$$N(\frac{1}{2}x_2 + x_3) = N(x + \frac{1}{2}).$$

Reversely the transition $3 \rightarrow 2$ is favoured by a surface energy which is weighted with $(+\frac{1}{2})$ by segments in state $\frac{1}{2}$ and $(+1)$ by those in state 0. Apart from this the surface energies in w_{12} and w_{32} should not differ requiring $\alpha_{12} = \alpha_{32}$. The surface energy for the transition $2 \rightarrow 1$ is weighted with $(+\frac{1}{2})$ for segments in state 0 and with $(-\frac{1}{2})$ for those in state 1, i.e. with $\frac{1}{2}(x_1 - x_3) = -x$. The remaining case $2 \rightarrow 3$ accordingly involves a factor $\frac{1}{2}(x_3 - x_1) = x$ with $\alpha_{21} = \alpha_{23}$. A positive volume energy β_{12} favours state $\frac{1}{2}$ in the transition $1 \rightarrow 2$, so that the inverse process should be weighted by its negative, $\beta_{21} = \beta_{12}$. Correspondingly for the transitions $3 \rightarrow 2$ and $2 \rightarrow 3$ we have $\beta_{23} = \beta_{32}$. Comparing Eqs. (21) with Eqs. (20) we get

$$\begin{aligned}\alpha_{12} &= \alpha_{32} = \alpha_{21} = \alpha_{23} = \frac{1}{2}, \\ \beta_{12} &= \beta_{21} = -\frac{1}{2} \left(\frac{\tilde{a}}{4} + \frac{\tilde{b}}{16} \right), \\ \beta_{23} &= \beta_{32} = -\frac{1}{2} \left(\frac{3\tilde{a}}{4} + \frac{15\tilde{b}}{16} \right).\end{aligned}\quad (22)$$

The transition rates are completely expressed by the Ginzburg-Landau parameters. This specification proceeded along the lines of the usual interpretation of the Ginzburg-Landau theory and should be still general enough to cover most of special applications.

We are left with four parameters including the common factor of proportionality in Eqs. (21), which are now fixed for numerical purposes. The free energy of a three-dimensional superconductor [8]

$$F = \int_V dv \left\{ \alpha_s(T) |\Delta|^2 + \frac{1}{2} \beta_s |\Delta|^4 + \frac{\hbar^2}{2m} |\nabla \Delta|^2 \right\}$$

leads to Eq. (15) by confining the volume V to the shape of a long, thin cylinder with radius π/k_F (k_F Fermi wavevector), corresponding to the mean electron distance in order of magnitude. Furthermore the maximum order parameter, at $T=0$, is normalized to unity defining

$$\psi \equiv \frac{\Delta}{\Delta_0}, \quad \Delta_0^2 = -\frac{\alpha_s(0)}{\beta_s} \equiv A k_F^3.$$

Integration over the sectional area of the cylinder yields Eq. (15) with

$$\begin{aligned}\xi_0 &= (\pi^3 A k_F)^{-1}, \quad \tilde{a} = \alpha_s(T), \\ \tilde{b} &= \frac{1}{2} \beta_s A k_F^3, \quad \tilde{c} = \hbar^2/2m.\end{aligned}$$

We chose the length Δx of a single segment as great with respect to the electron distance, particularly

comparable with the temperature independent coherence length $\xi_{GL}(\Delta x = \mu \xi_{GL})$. Thus the fluctuations in space along the scale of the order parameter are reproduced whereas the microscopic variations are averaged. With the inverse reduced temperature $\tau = T_c/T$ and a constant ν the transition rates of Eqs. (21) inserting Eqs. (22) read

$$\begin{aligned}w_{12} &= \nu \exp \left\{ - \left(\frac{a}{4} + \frac{b}{16} \right) + c \left(x + \frac{1}{2} \right) \right\}, \\ w_{21} &= \nu \exp \left\{ \frac{a}{4} + \frac{b}{16} - cx \right\}, \\ w_{23} &= \nu \exp \left\{ - \left(\frac{3a}{4} + \frac{15b}{16} \right) + cx \right\}, \\ w_{32} &= \nu \exp \left\{ \frac{3a}{4} + \frac{15b}{16} - c \left(x - \frac{1}{2} \right) \right\},\end{aligned}\quad (23)$$

and [8]

$$\begin{aligned}a &= \frac{\beta \Delta x}{2 \xi_0} \tilde{a} = 2.71 \mu (1 - \tau), \\ b &= \frac{\beta \Delta x}{2 \xi_0} \tilde{b} = 1.35 \mu \tau, \\ c &= \frac{\beta}{\xi_0 \Delta x} \tilde{c} = 2.96 \tau / \mu.\end{aligned}$$

The behaviour of the system strongly changes with parameter μ , the ratio of segment length to coherence length. Two values 1.2, 2 will be assigned to μ . The constant ν only determines the time scale and is of no importance.

Numerical Results

The equations of motion Eqs. (12), (13) determine from given initial values the time evolution of the mean order parameter, Eq. (10), and of its mean square, Eq. (11), both measured relative to the state $\psi_i = \frac{1}{2}$. The motion of the respective points in a two-dimensional plot is confined to a triangular area $-\frac{1}{2} \leq \bar{x} \leq +\frac{1}{2}$, $0 \leq \bar{y} \leq \frac{1}{4}$, corresponding to the condition $\bar{x}_1 + \bar{x}_2 + \bar{x}_3 = 1$, and ends in (x_e, y_e) , the stable or unstable equilibrium positions (Figure 1). The latter is decided by comparing the free energy.

Equations (12), (13) can be rewritten as

$$\dot{\mathbf{a}} = A \mathbf{a} + \mathbf{b}, \quad \mathbf{a} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}, \quad (24)$$

where the vector \mathbf{b} with components

$$b_1 = \frac{1}{2} (-w_{21} + w_{23}), \quad b_2 = \frac{1}{4} (w_{21} + w_{23}),$$

and the matrix \mathbf{A} with

$$\begin{aligned} A_{11} &= -\frac{1}{2}(w_{12} + w_{32}), \\ A_{12} &= w_{12} - w_{32} + 2(w_{21} - w_{23}), \\ A_{21} &= \frac{1}{4}(w_{12} - w_{32}), \\ A_{22} &= -(w_{21} + w_{23} + \frac{1}{2}(w_{12} + w_{32})) \end{aligned}$$

still depend on x . Equation (24) is solved numerically for various initial values. Equilibrium implies $\dot{\mathbf{a}} = 0$ yielding

$$x_e = \frac{\sinh(2cx_e - a - b)}{\exp\{\frac{1}{2}(a + c) + \frac{1}{2}b\} + 2 \cosh(2cx_e - a - b)}, \quad (25)$$

$$y_e = \frac{1}{2} \frac{w_{21} + w_{23} + x_e(w_{12} - w_{32})}{w_{12} + w_{32} + 2(w_{21} + w_{23})}. \quad (26)$$

Equation (25) is a transcendental equation for x_e , identical with the usual selfconsistency condition of the mean-field approximation, if one performs the sum over states in the partition function

$$Z = \sum_{\{\psi_k\}=0, \frac{1}{2}, 1} \exp \left\{ -\frac{\beta \Delta x}{\xi_0} \sum_{i=1}^N f(\psi_{i+1}, \psi_i) \right\}$$

and in the expression

$$\bar{\psi} = \frac{1}{Z} \sum_{\{\psi_k\}=0, \frac{1}{2}, 1} \psi_k \exp \left\{ -\frac{\beta \Delta x}{\xi_0} \sum_{i=1}^N f(\psi_{i+1}, \psi_i) \right\}$$

by using Eq. (18) with x fixed at the equilibrium value $x_e = \bar{\psi} - \frac{1}{2}$. To test for the stability of the equilibrium we consider on the one hand in the same approximation the free energy difference relative to the normal state

$$\begin{aligned} \Delta f(x_e) &= -\frac{1}{\beta N} \ln Z \\ &= -\frac{1}{\beta} \left[-2c \left(x_e + \frac{1}{2} \right)^2 \right. \\ &\quad \left. + \ln \left(1 + \exp \left\{ -\frac{a}{2} - \frac{b}{8} + 2c \left(x_e + \frac{1}{4} \right) \right\} \right) \right. \\ &\quad \left. + \exp \{ -2(a + b - 2cx_e) \} \right]. \quad (27) \end{aligned}$$

On the other hand we calculate the relaxation rates near the equilibrium positions. To linear order we get

$$\dot{\mathbf{a}} = \mathbf{A}_e(\mathbf{a} - \mathbf{a}_e) \quad (28)$$

with

$$\mathbf{A}_e = \begin{pmatrix} x_e A'_{11} + y_e A'_{12} + b'_1 + A_{11} & A_{12} \\ x_e A'_{21} + y_e A'_{22} + b'_2 + A_{21} & A_{22} \end{pmatrix},$$

where the prime denotes differentiation with respect to the only variable \bar{x} putting everywhere $\bar{x} = x_e$. Inserting

$$\mathbf{a} - \mathbf{a}_e = \mathbf{a}_0 e^{\lambda t}$$

into Eq. (28) the relaxation rates are determined by

$$\lambda_{\pm} = \frac{1}{2} \text{Tr } \mathbf{A}_e \pm [\frac{1}{4}(\text{Tr } \mathbf{A}_e)^2 - \text{Det } \mathbf{A}_e]^{1/2}. \quad (29)$$

Stability requires $\lambda < 0$. A vanishing relaxation rate characterizes critical slowing down, indicating that great fluctuations render the Gauß approximation questionable and limiting the considerations of averaged quantities.

In Figs. 1, 2 the trajectories of the system, Eq. (24), are plotted in the (\bar{x}, \bar{y}) -plane showing the time behaviour of the average occupation of the three levels $\psi = 0, \frac{1}{2}, 1$. The boundaries of the domain are set by a vanishing population in one of these levels, $\bar{x}_1 = 0$, $\bar{x}_3 = 0$ or $\bar{x}_2 = 0$. Of course they are never crossed by the path of the system. The overall attitude shows a movement towards the final equilibrium point. However, it is interesting that not the shortest path is taken. The first development from an initial point on $\bar{x}_2 = 0$ ($\bar{y} = \frac{1}{4}$) consists in filling up the level $\psi = \frac{1}{2}$ together with depleting one of the extreme levels, which is shown by the rather straight movement towards the nearest one of the boundaries $\bar{x}_1 = 0$ or $\bar{x}_3 = 0$. Thereby it even occurs that a level is first depleted which later is supplemented to reach the final equilibrium (e.g. $\mu = 1.2$, $\tau = 4$ in Figure 1). Once the system has entered the neighbourhood of a lower boundary it moves along the boundary. This attraction by the lower boundaries reflects a decreasing surface energy for increasing population of only two levels with minimum difference in height $\Delta\psi = \frac{1}{2}$. Furthermore the preferred occupation of the central level seems to be also a consequence of the exclusion of direct transitions $\psi_i = 0 \leftrightarrow \psi_i = 1$. Running a glance over the social analogy of Weidlich [4] the addition of a third opinion between both extremes leads temporarily to a very pronounced identification with this opinion even to the loss of the latter stabilizing general attitude. For the Ginzburg-Landau model we conclude that at first the fluctuations along the chain are suppressed regardless of the final state and then the equilibrium state is approached. The former is produced by the spin-like interaction term in the Ginzburg-Landau functional, as seen in the more pronounced behaviour

for the smaller values μ , the latter takes the volume energy into account.

The equilibrium quantities corresponding to the final points e_i in the paths of the system are shown in Figures 3, 4. The order parameter increases with decreasing temperature showing a greater slope near the transition temperature T_c of the Ginzburg-Landau theory which is accompanied by a slight change in the slope of the free energy, by a peak in

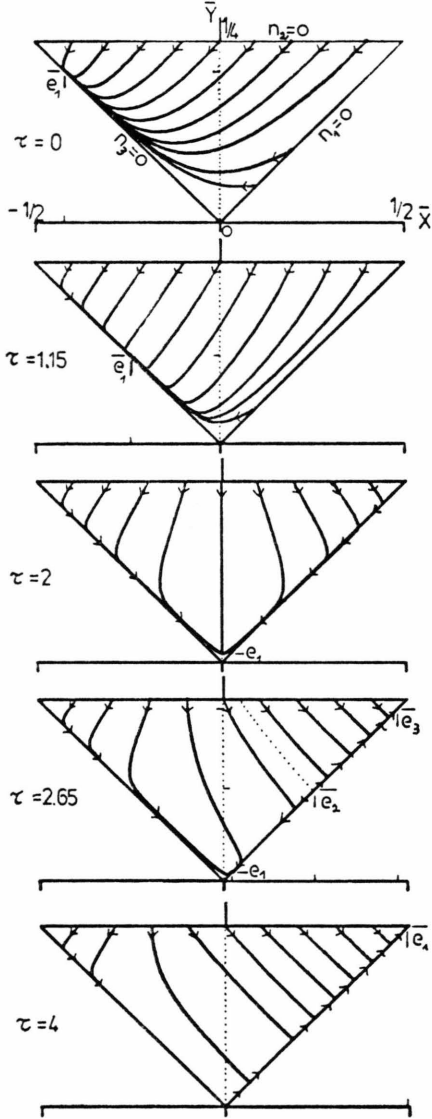


Fig. 1. Time evolution for $\mu = 1.2$ and various $\tau = T_c/T$ with \bar{x} the mean order parameter, \bar{y} its mean square, both relative to the state $\psi = \frac{1}{2}$; e_i denoting the equilibrium points.

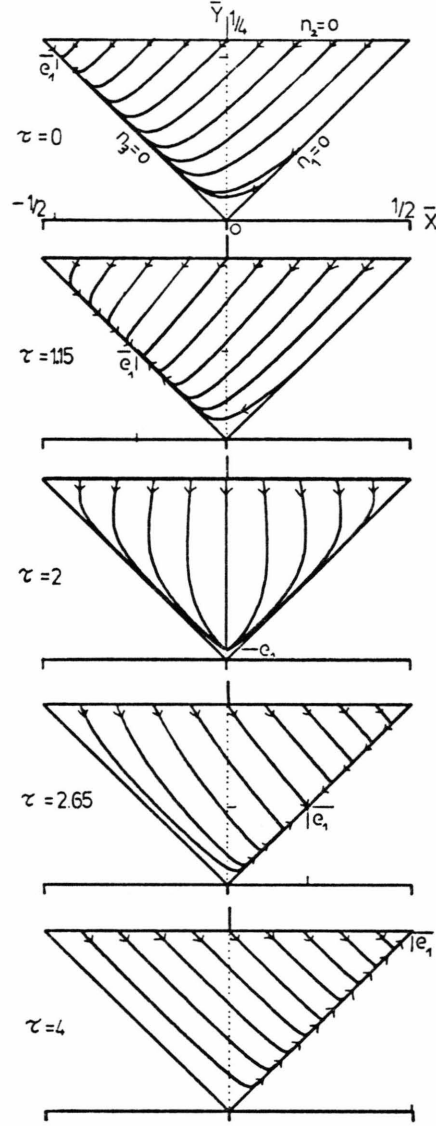


Fig. 2. Time evolution for $\mu = 2$; (compare Figure 1).

the fluctuations along the chain and by a smaller relaxation rate for the deviation from equilibrium in the λ_+ -direction. The latter is one of the eigenvectors of the matrix A_e with eigenvalues λ_{\pm} , Equation (29). The positive sign refers to a direction almost parallel to the boundary and characterizes near equilibrium the path the system prefers in most cases (Figures 1, 2). When coming far from equilibrium the system generally follows the direction with slow relaxation as soon as the boundary is reached.

In decreasing the temperature one passes through a second ordering driven by the surface energy which appears like a first order phase transition for the stronger coupling $\mu = 1.2$. For $\mu = 2$ fluctuations are very pronounced rising to a sharp maximum and the relaxation, λ_+ , is just slowing down. At this temperature $\mu = 1.2$ yields bifurcated states one of which is unstable, i.e. it belongs to the higher free energy and it moves away exponentially with a positive relaxation rate λ_+ . Because of the μ -dependence of the surface term the system at this particular ordering temperature behaves more like a spin system for decreasing μ , though the characteristic properties of the instability are exaggerated by the mean-field approximation. Increasing μ re-

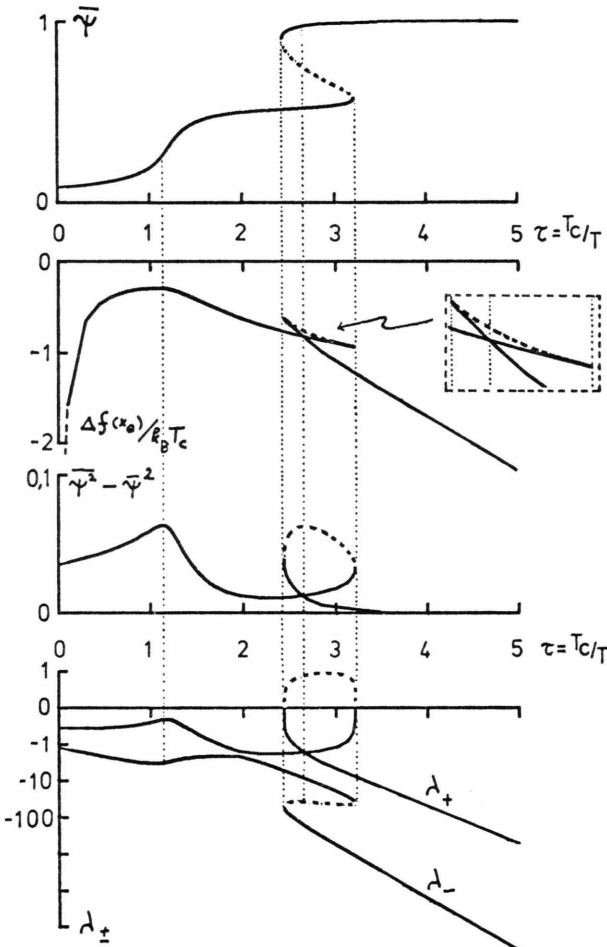


Fig. 3. Mean order parameter $\bar{\psi}$, its mean square $\bar{\psi}^2 - \bar{\psi}^2$, free energy difference $\Delta f(x_e)$ relative to the normal state and relaxation rates λ_{\pm} vs. inverse reduced temperature τ in the equilibrium positions for $\mu = 1.2$.

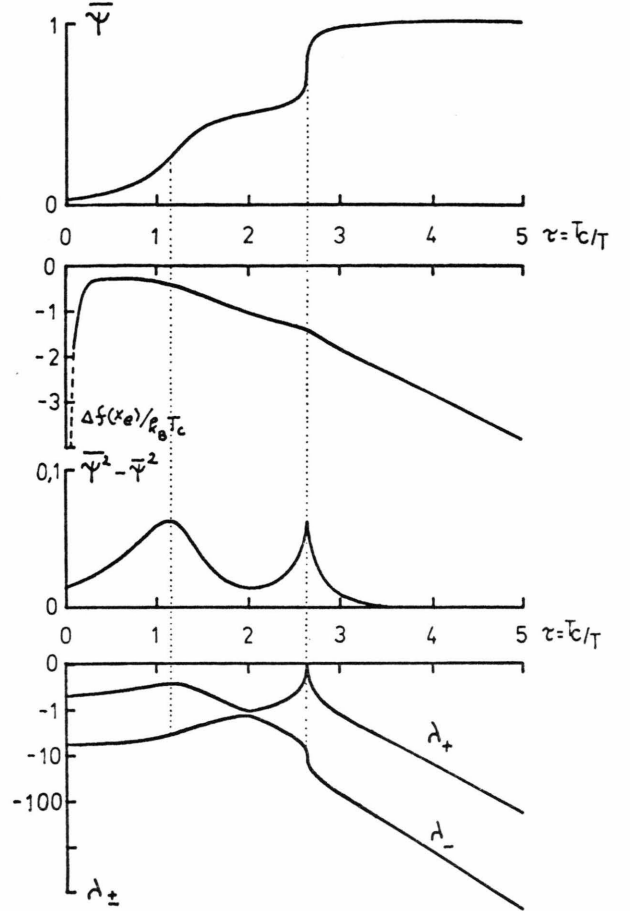


Fig. 4. Plots corresponding to Fig. 3 for $\mu = 2$.

duces the critical behaviour. It shows that the model depends sensitively on the choice of the segment length within the range which in a natural way is fixed by the Ginzburg-Landau coherence length.

These preliminary studies lead to a rough picture of the dynamical behaviour of a Ginzburg-Landau model suggesting further investigations beyond the mean-field approximation especially with respect to the probability distribution in the neighbourhood of criticality. The calculations show that without assuming the Josephson coupling we get a picture similar to granular superconductors with two distinct transitions — one being the single-particle transition of the individual grain and the other reflecting the collective ordering of the grains — though matters may become complicated depending on the choice of the length scale μ .

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